Section 1.1 Systems of Linear Equations

Recall: A linear equation in the variables $x_1, \dots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

$b, a_i \in \mathbb{R}$ or $\mathbb{C}$, $n \in \mathbb{N}$

Defn
A system of linear equations is a collection of linear equations using the same variables.

Ex: ①
$$\begin{aligned} x + y &= 5 \\ 2x - y &= 7 \end{aligned}$$

or ②
$$\begin{aligned} x_1 + \sqrt{2}\, x_2 - .5\, x_3 &= 7 \\ 2x_1 \qquad\quad + 3x_3 &= -1 \end{aligned}$$

Defn
A solution of the system is a list $(s_1, s_2, \dots, s_n)$ of numbers that make each equation true when values $s_1, s_2, \dots, s_n$ are plugged in for $x_1, \dots, x_n$ (respectively)

Notice that $(5.125, 0, -3.75)$ is a solution to ②. (It isn't nec. the only solution!)

Note: solving a system of two linear equations in 2 variables amounts to finding where two lines intersect.
However, with more variables and more equations, we need new tools to handle the abstraction.

We can encode the essential information of a linear system in a rectangular array called a matrix

Ex: Given
$$\begin{aligned} x_1 - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

we can put this info into a matrix:

$$\begin{array}{c} \text{coeff} \quad \begin{array}{ccccc} x_1 & x_2 & x_3 & = & \# \end{array} \\ \begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \end{array}$$

This is called the augmented matrix of the system.

The matrix of coefficients would be the first 3 columns.

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix}$$

which encodes info about coeff on x_1, x_2, x_3.

Defn: The size of a matrix tells how many rows and colums it has.

An $m \times n$ matrix has m rows and n columns.

Ex: The augmented matrix before ~~has~~ is a 3×4 matrix.

Question: what does this augmented matrix do for us?

Answer: We can use it to solve its associated system of linear eqs!

Ex:

$$\begin{array}{ll} 1 & x_1 \qquad\quad - 3x_3 = 8 \\ 2 & 2x_1 + 2x_2 + 9x_3 = 7 \\ 3 & \qquad\quad x_2 + 5x_3 = -2 \end{array} \qquad \begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

By adding multiples of equation 1 to the others, we can eliminate x_1 from everywhere but eq 1:

$$\begin{array}{rl} -2\text{ Eq }1 & -2x_1 \qquad\quad + 6x_3 = -16 \\ + \quad \text{Eq }2 & 2x_1 + 2x_2 + 9x_3 = 7 \\ \hline & 0x_1 + 2x_2 + 15x_3 = -9 \end{array}$$

we can replace E2 with this new Eq. (Replacement.)

Now

$$\begin{array}{lr} x_1 \qquad\quad - 3x_3 = 8 & ① \\ 2x_2 + 15x_3 = -9 & ② \\ x_2 + 5x_3 = -2 & ③ \end{array} \qquad \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 2 & 5 & -2 \end{bmatrix}$$

Now divide Eq 2 by 2 to get the coeff on $x_2 = 1$: (Scaling)

$$\begin{aligned} x_1 \qquad\quad -3x_3 &= 8 \\ x_2 + \tfrac{15}{2}x_3 &= -\tfrac{9}{2} \\ x_2 + 5x_3 &= -2. \end{aligned}$$

Now eliminate x_2 from Eq 3:

$$\begin{array}{rl} -\text{ Eq }2 & -x_2 - \tfrac{15}{2}x_3 = \tfrac{9}{2} \\ + \text{ Eq }3 & + \quad x_2 + 5x_3 = -2 \\ \hline & 0x_2 - \tfrac{5}{2}x_3 = \tfrac{5}{2} \end{array}$$

Replace:

$$\begin{aligned} x_1 \qquad\quad -3x_3 &= 8 \\ x_2 + \tfrac{15}{2}x_3 &= -\tfrac{9}{2} \\ -\tfrac{5}{2}x_3 &= \tfrac{5}{2} \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & \tfrac{15}{2} & -\tfrac{9}{2} \\ 0 & 0 & -\tfrac{5}{2} & \tfrac{5}{2} \end{bmatrix}$$

We have a triangular form!

what just happened?

We eliminated variables in lower equations so we get an equation that says $x_3 = \#$ (or x_n). If we have this, we can plug into Eq 2, which only had variables x_2 & x_3 to get $x_2 = \#$, etc.

<u>Defn</u>: A solution is <u>consistent</u> if it has at least one solution.

<u>Fact</u>: If you have a row that looks like

$$[0 \ 0 \cdots 0 \ b] \quad b \neq 0$$

then the system isn't consistent (or <u>inconsistent</u>), as the row says

$$0x_1 + 0x_2 + \cdots + 0x_n = b$$

$$0 = b.$$

which isn't true.

We can streamline this solving process by using only the matrix. To do this, we'll need the following:

Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row
2. (Interchange) Interchange 2 rows
3. (Scaling) Multiply all entries in a row by a nonzero constant

helpful if your first eq doesn't have an x_1, etc.

<u>Ex</u>: Solve the following system:

$$2x_1 - y + 3z = 6$$
$$3x - 5y + 4z = 7$$
$$2x + y + z = -2$$

$(-3, 0, 4)$

$$3x_1 + 2x_2 + x_3 = 3$$
$$x_1 - 3x_2 + x_3 = 4$$
$$-6x_1 - 4x_2 - 2x_3 = 1$$

(no soln)

Hint! First, make the augmented matrix.
second, using row operations, try to get into "triangular form"

<u>You do</u>:
$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \xrightarrow{4 \times R_1 + R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \xrightarrow[\text{by } 2]{\text{divide } R_2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\xrightarrow{3 \times R_2 + R_3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

consistent! $x_3 = 3$.

$x_2 - 4x_3 = 4$

$x_2 - 4(3) = 4$

$x_2 = 16$.

$x_1 - 2x_2 + x_3 = 0$

$x_1 - 2(16) + 3 = 0$

$x_1 = 29$

Section 1.2

Defn: A <u>leading entry of a row</u> is the leftmost nonzero entry.

Defn: A matrix is in (row) <u>echelon form</u> if

1. All nonzero rows are above any zero rows
2. Each leading entry of a row is in a column to the right of the leading entry above it
3. All entries in a column below a leading entry are zero.

A matrix is row echelon form is in reduced row echelon form (RREF) if also

4. The leading entry in each row is 1
5. Each leading 1 is the only nonzero entry in its column.

<u>Exs</u>:

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 & 0 & 0 \\ 4 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Row EF.   neither   Red REF

<u>Theorem</u>: Each matrix is row equivalent to one and only one reduced echelon matrix. That is, the RREF of a matrix is <u>unique</u>!

<u>proof</u>: Maybe later in the semester.

Defn: A <u>pivot position</u> in a matrix A is the location/position in A that corresponds to a leading 1 in the reduced row echelon form of A.
A <u>pivot column</u> is a column of A that contains a pivot position.

<u>Ex</u>:

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix} \xrightarrow[-4R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & -3 & -12 & -18 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

$$\xrightarrow{\text{div by } -3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -3 & -6 \end{bmatrix} \xrightarrow{\text{div by } -3} \begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Ex: Above:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

Detailed Algorithm in book Sec 1.2 : Read it!

Ex:

Solutions to Linear Systems

The row reduction algorithm leads to an explicit solution to the system.

Defn: The variables corresponding to pivot columns are called basic variables. All other variables are called free variables.

Ex:
$$\begin{array}{ccccc} & & & & \\ \end{array}\begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$
$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

x_3 is free since column 3 isn't a pivot column

Free = any value can be chosen

Note: RREF puts each basic variable in only one equation (as pivot columns have a single 1 and all zeros)

So we can describe the solution to this system: Solve basic variables in terms of free variables:

$$x_1 + 2x_3 = 4$$
$$x_2 + 5x_3 = 1$$
$$x_4 = 6$$

solution:
$$\begin{cases} x_1 = 4 - 2x_3 \\ x_2 = 1 - 5x_3 \\ x_3 \text{ free} \\ x_4 = 6 \end{cases}$$

Ex: Find general solution of the system whose augmented matrix is

$$\begin{bmatrix} 0 & 1 & -2 & 3 \\ 1 & -3 & 4 & -6 \end{bmatrix}$$

Hint: Top left must be a pivot position! Then get into RREF.

$$\rightsquigarrow \begin{bmatrix} 1 & -3 & 4 & -6 \\ 0 & 1 & -2 & 3 \end{bmatrix} \underset{3R_2+R_1}{\overset{\text{RREF}}{\rightsquigarrow}} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{bmatrix} \rightsquigarrow \begin{matrix} x_1 - 2x_3 = 3 \\ x_2 - 2x_3 = 3 \end{matrix}$$

pivot col   pivot col   free

General solution:
$$\begin{cases} x_1 = 2x_3 + 3 \\ x_2 = 2x_3 + 3 \\ x_3 \text{ free.} \end{cases}$$

Q: when is there a soln, and when is it unique?

Recall ~~Theorem~~: A linear system is consistent if and only if it has no row of the form $[0 \cdots 0\ b]$ $b \neq 0$ in its ref augmented matrix. That is, if its rightmost column is not a pivot column.

Theorem: If a linear system is consistent, then the solution set contains either (i) A unique solution, when there are no free variables or (ii) infinitely many solutions, if there is at least one free variable.