RESEARCH STATEMENT

1. Introduction

Informally, a loop is a "nonassociative group". More precisely, a *loop* is a set $(Q, \cdot)$ with a binary operation $\cdot$ such that (i) $(Q, \cdot)$ is a *quasigroup*, that is, for each $a, b \in Q$, there exist unique $x, y \in Q$ with $ax = b$ and $ya = b$, and (ii) there exists an identity element $1 \in Q$ such that $1x = x1 = x$ for all $x \in Q$. Standard references for loop theory are [3, 25].

Loops (and quasigroups) are not just generalizations for the sake of generalization. They appear quite naturally in many parts of mathematics. Historically, loop theory is most closely connected with combinatorics, particularly latin squares. Indeed, normalized latin squares are precisely the multiplication tables of finite loops. Loops are also the coordinatizing structures for 3-nets, which are close relatives of projective planes.

Loops arise naturally in physics, particularly in special relativity. The set $\{\mathbf{v} \in \mathbb{R}^3 \mid |\mathbf{v}| < c\}$ of all relativistic velocity vectors forms a loop where the operation is Einstein's velocity addition formula. This is an example of a *Bruck loop* [26]. Another example of a Bruck loop is given on the set $H^+(n, \mathbb{C})$ of all $n \times n$ positive definite Hermitian matrices by the polar decomposition. Given two such matrices A and B, let $AB = PU$ be the polar decomposition where P is positive definite Hermitian and U is unitary. Defining $A \circ B = P$ gives $H^+(n, \mathbb{C})$ the structure of a Bruck loop [17].

An even better known class of loops is typified by the sphere S^7 under octonion multiplication, or more generally, the set of all nonzero octonions under multiplication. Even more generally, one can take the set of all invertible elements in an alternative ring. All of these loops are examples of *Moufang loops*, about which more is known then perhaps any other type of loop. Moufang loops are closely related to groups with *triality* (and in fact, these notions are essentially the same). The deepest questions in the theory of Moufang loops are often resolved by formulating them in group theoretic terms and using the corresponding powerful tools of group theory. For instance, all finite simple Moufang loops are classified because finite simple groups with triality are classified [21].

2. Background

In a loop Q, the left and right translations by $x \in Q$ are defined by $yL_x = xy$ and $yR_x = yx$ respectively. We have the following sets and associated groups of interest:

- *left and right translation by x* $\quad yL_x = xy \quad yR_x = yx$,
- *left section of Q* $\quad L_Q = \{L_x \mid x \in Q\}$,
- *right section of Q* $\quad R_Q = \{R_x \mid x \in Q\}$,
- *multiplication group of Q* $\quad \mathrm{Mlt}(Q) = \langle L_Q, R_Q\rangle$,
- *inner mapping group of Q* $\quad \mathrm{Inn}(Q) = \{\theta \in \mathrm{Mlt}(Q) \mid 1\theta = 1\}$.

2010 *Mathematics Subject Classification.* 20N05.

Moufang loops, which are easily the most studied class of loops, are defined by the identity $(xy)(zx) = x((yz)x)$ (or other identities equivalent to this). A loop Q is automorphic if every inner mapping of Q is an automorphism of Q (*i.e.* Inn(Q)$\leq$ Aut(Q)). The variety of automorphic loops include groups and commutative Moufang loops. Automorphic loops were first studied by Bruck and Paige [4].

My current research focuses on two main ideas: studying loops that generalize Moufang loops and *commutative* automorphic loops; and constructions for different varieties of loops.

3. Research Summary

3.1. **Γ-Loops.** Let G be a uniquely 2-divisible group, that is, a group in which the map $x \mapsto x^2$ is a bijection. On G we define two new binary operations as follows:

$$x \oplus y = (xy^2x)^{1/2}\,, \tag{3.1}$$

$$x \circ y = xy[y,x]^{1/2}\,. \tag{3.2}$$

Here $a^{1/2}$ denotes the unique $b \in Q$ satisfying $b^2 = a$ and $[y,x] = y^{-1}x^{-1}yx$. Then it turns out that both $(G,\oplus)$ and $(G,\circ)$ are loops with neutral element 1. The loop $(G,\oplus)$ is well known, with the basic ideas dating back to Bruck [3] and Glauberman [7]. $(G,\oplus)$ is an example of a *Bruck loop*, that is, it satisfies the following identities

$$(x \oplus (y \oplus x)) \oplus z = x \oplus (y \oplus (x \oplus z)), \tag{Bol}$$

$$(x \oplus y)^{-1} = x^{-1} \oplus y^{-1}. \tag{AIP}$$

Bruck loops are *power-associative*, which informally means that integer powers of elements can be defined unambiguously. Further, powers in G and powers in $(G,\oplus)$ coincide.

$(G,\circ)$ turns out to live in a variety of loops which we will call Γ-*loops*,

Definition 3.1. *A loop $(Q,\cdot)$ is a Γ-loop if the following hold*

- (Γ_1) *Q is commutative.*
- (Γ_2) *Q has the automorphic inverse property (AIP): $\forall x,y \in Q$, $(xy)^{-1} = x^{-1}y^{-1}$.*
- (Γ_3) *$\forall x \in Q$, $L_xL_{x^{-1}} = L_{x^{-1}}L_x$.*
- (Γ_4) *$\forall x,y \in Q$, $P_xP_yP_x = P_{yP_x}$ where $P_x = R_xL^{-1}_{x^{-1}} = L_xL^{-1}_{x^{-1}}$.*

Γ-loops include as special cases two classes of loops: commutative semiautomorphic IP loops [20, 10] (see §3.2) and commutative automorphic loops [14, 13, 15, 5]. Our first goal was showing Γ-loops are power-associative:

Theorem 3.2. *Γ-loops are power associative.*

As a consequence, powers of elements in G, $(G,\circ)$ and $(G,\oplus)$ all coincide.

Jedlička, Kinyon and Vojtěchovský [14] showed that starting with a uniquely 2-divisible commutative automorphic loop $(Q,\circ)$, one can define a Bruck loop $(Q,\oplus')$ on the same underlying set Q by

$$x \oplus' y = (x^{-1}\backslash_\circ(y^2 \circ x))^{1/2}\,. \tag{3.3}$$

Here $a\backslash_\circ b$ is the unique solution c to $a \circ c = b$. This results also extends to Γ-loops [9]. This gives us a functor $\mathcal{B} : \underline{\Gamma\mathrm{Lp}}_{1/2} \rightsquigarrow \underline{\mathrm{BrLp}}_{1/2}$ from the category $\underline{\Gamma\mathrm{Lp}}_{1/2}$ of uniquely 2-divisible Γ-loops to the category $\underline{\mathrm{BrLp}}_{1/2}$ of uniquely 2-divisible Bruck loops. Our second main result is the construction of an inverse functor $\mathcal{G} : \underline{\mathrm{BrLp}}_{1/2} \rightsquigarrow \underline{\Gamma\mathrm{Lp}}_{1/2}$, that is, $\mathcal{G} \circ \mathcal{B}$ is the identity functor on $\underline{\Gamma\mathrm{Lp}}_{1/2}$ and $\mathcal{B} \circ \mathcal{G}$ is the identity functor on $\underline{\mathrm{BrLp}}_{1/2}$.

Theorem 3.3. *$\underline{\Gamma\mathrm{Lp}}_{1/2}$ and $\underline{\mathrm{BrLp}}_{1/2}$ are categorically isomorphic.*

Finite Bruck loops of odd order are known to have many remarkable properties, all found by Glauberman [7, 8]. For instance, they satisfy Lagrange's Theorem, the Odd Order Theorem, the Sylow and Hall Existence Theorems and finite Bruck p-loops (p odd) are centrally nilpotent. Using the isomorphism of the categories $\underline{\Gamma\mathrm{Lp}}_{1/2}$ and $\underline{\mathrm{BrLp}}_{1/2}$, we immediately get the same results for Γ-loops of odd order.

Originally, our motivation was to answer an open problem of Jedlička, Kinyon and Vojtěchovský [14], dealing with the existence of Sylow and Hall subgroups in finite commutative automorphic loops. The authors showed that a solution would follow from an answer in the odd order case [14]. Using this and the new isomorphism, the Sylow and Hall Theorems for Γ-loops of odd order are answered in the affirmative, in a more general way than was originally posed. Further, the proofs of the Odd Order Theorem and the nontriviality of the center of finite Γ-p-loops (p odd) are much simpler than the proofs in [14] and [15] for commutative automorphic loops.

3.2. **Semiautomorphic Inverse Property Loops.** In general, the inner mappings of a nonassociative loop are not automorphisms of the loop (except, by definition, in the class of automorphic loops). However, in some of the various classes of loops which are commonly studied, the action of the inner mapping group still preserves some of the loop structure. For example, every inner mapping θ of a Moufang loop Q is a *semiautomorphism*, that is, $1\theta = 1$ and

$$(xyx)\theta = x\theta \cdot y\theta \cdot x\theta$$

for all $x, y \in Q$. (Since Moufang loops are *flexible*, that is, $(xy)x = x(yx)$ for all x, y, we may write xyx unambiguously.)

Steiner loops, which arise from Steiner triple systems, are loops satisfying the identities $xy = yx$, $x(yx) = y$. Every inner mapping θ of a Steiner loop is also (trivially!) a semiautomorphism: $(xyx)\theta = y\theta = x\theta \cdot y\theta \cdot x\theta$.

We focused on this property of inner mappings to study a class of loops generalizing both Moufang loops and Steiner loops.

Definition 3.4. *A loop Q is said to be a* semiautomorphic, inverse property loop *(or just semiautomorphic IP loop) if*

(1) Q is flexible, that is, $(xy)x = x(yx)$ for all $x, y \in Q$;
(2) Q has the inverse property *(IP), that is, for each $x \in Q$, there exists $x^{-1} \in Q$ such that $x^{-1}(xy) = y$ and $(yx)x^{-1} = y$ for all $y \in Q$:*
(3) Every inner mapping is a semiautomorphism, that is, for each $\theta \in Inn(Q)$, $x\theta \cdot y\theta \cdot x\theta = (x \cdot y \cdot x)\theta$ for all $x, y \in Q$.

If θ is a semiautomorphism of a flexible loop Q, then for all $x \in Q$, $x\theta = (xx^{-1}x)\theta = x\theta \cdot x^{-1}\theta \cdot x\theta$, and cancelling gives $1 = x\theta \cdot x^{-1}\theta$. Thus if we define the inversion map $J : Q \to Q$ by $xJ = x^{-1}$, we have $\theta^J = \theta$ for any semiautomorphism.

It follows that any semiautomorphic IP loop is an example of a variety of loops which have already appeared in the literature called "RIF loops" (RIF = **R**espects **I**nverses and

Flexible). General RIF loops were introduced in [18] and commutative RIF loops were studied in [20]. Recalling that a loop is *diassociative* if any subloop generated by at most 2 elements is associative, we have the following, which follows from the main result of [18].

Proposition 3.5. *(*[18]*). Every semiautomorphic IP loop is diassociative.*

Our first main result, is the converse of our observation that every semiautomorphic IP loop is a RIF loop. We state this as the following characterization:

Theorem 3.6. *Let Q be a loop. The following are equivalent.*

(1) Q is a semiautomorphic IP loop;
(2) Q is a flexible IP loop such that $\theta^J = \theta$ for all $\theta \in Inn(Q)$.

The *commutant* of a loop Q is the set $C(Q) = \{a \in Q \mid ax = xa \ \forall x \in Q\}$. In general, the commutant of a loop is not a subloop, although it is known to be so in certain cases, such as for Moufang loops.

Theorem 3.7. *The commutant of a semiautomorphic IP loop is a subloop.*

In proving Theorem 3.7, we show that for any $a \in C(Q)$, a^2 is a Moufang element, that is, $a^2 \cdot xy \cdot a^2 = a^2x \cdot ya^2$ for all x, y. This immediately gives us that for each $a \in C(Q)$, $a^6 \in Z(Q)$, where $Z(Q)$ denotes the center of Q. This simultaneously generalizes two results: that in a Moufang loop, the cube of any commutant element is central [3], and that in a commutative semiautomorphic IP loop, the sixth power of any element is central [20].

We have two constructions for creating semiautomorphic loops. There is a well-known doubling construction of Chein which builds nonassociative Moufang loops from nonabelian groups. The construction itself makes sense even when one starts with a loop instead of a group. It turns out that if one applies the construction to a semiautomorphic IP loop, the result is another semiautomorphic IP loop [10]. In particular, this allows us to construct nonMoufang, nonSteiner, semiautomorphic IP loops by starting with nonassociative Moufang loops.

Our second construction is based on another doubling technique of de Barros and Juriaans. It was already noted (without human proof) that applying the de Barros-Juriaans construction to a group gives what we are now calling a semiautomorphic IP loop. Again we have that just as with the Chein construction, starting with a semiautomorphic IP loop in the de Barros-Juriaans construction yields another semiautomorphic IP loop [10]. Also, if we start with a semiautomorphic IP loop, apply the de Barros-Jurrians construction and then apply the Chein construction to the result, we end up with the same loop up to isomorphism as if we had applied the Chein construction twice [10].

3.3. **Simple Right Conjugacy Closed Loops.** A loop Q is a *right conjugacy closed loop*, or RCC loop, if $R_x^{-1}R_yR_x \in R_Q$, or equivalently $(xy)z = (xz) \cdot z\backslash(yz)$ for all $x, y, z \in Q$. Most of the literature on the variety of conjugacy closed loops deals with left conjugacy closed loops, but given an LCC loop Q, $(Q, \circ)$ is a RCC loop with $x \circ y = yx$. General theory of LCC loops can be found in [2, 22, 6]. However, there is little discussion of simple LCC loops.

A subloop N of a loop Q is *normal* if it is invariant under $\mathrm{Inn}(Q)$. Q is *simple* if the only normal subloops are the trivial subloops Q and $\{1\}$. As with groups, simple loops in a particular variety form the basic building blocks of the variety. Using general linear groups, we construct simple RCC loops.

First, let $f(x) = x^2 - rx + s$ be irreducible in $\mathbb{F}_q[x]$. For each $b \in \mathbb{F}_q$, define $M_{f(0,b)} = \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right)$ and for $a \neq 0$, $M_{f(a,b)} = \begin{pmatrix} r-b & \frac{s-br-b^2}{-a} \\ a & b \end{pmatrix}$. Then,

Lemma 3.8. *Let $f(x) = x^2 - rx + s$ be irreducible in $\mathbb{F}_q[x]$. The conjugacy class of all matrices in $GL(2,q)$ with characteristic polynomial $f(x)$ is precisely the set $\{M_{f(a,b)} \mid a, b \in F_q\}$.*

Theorem 3.9. *Let $f(x) = x^2 - rx + s$ be irreducible in $\mathbb{F}_q[x]$. Let $Q = \mathbb{F}_q^2$, written as a set of row vectors. Define a binary operation $\circ_f$ on Q by $[a,b] \circ_f [c,d] = [a,b]M_{f(c,d)}$. Then $(Q, \circ_f)$ is an RCC loop.*

Moreover,

Theorem 3.10. *If the trace of $M_{f(a,b)} \neq 0$ for every $M_{f(a,b)}$, then $(Q, \circ_f)$ is simple. Else, $(Q, \circ_f)/Z(Q, \circ_f)$ is simple.*

4. Current & Future Research

4.1. **Automorphic loops.** Recently, automorphic loops were shown to satisfy the Odd Order Theorem, that is, every finite automorphic loop of odd order is solvable; Cauchy's Theorem is known to hold; and automorphic loops satisfy the elementwise Lagrange Theorem, that is, the order of an element divides the order of the loop [19]. There are still several basic open problems (all answered in the commutative case, §3.1):

Problem 4.1. *Let Q be a finite automorphic loop.*

(i) Let $S \leq Q$. Then does $|S|$ divide $|Q|$?
(ii) For each prime p dividing $|Q|$, does Q have an element of order p?
(iii) For each prime p dividing $|Q|$, does Q have a Sylow p-subgroup?
(iv) If Q is solvable and if π is a set of primes, does Q have a Hall π-subloop?

However, the main open problem in the theory of automorphic loops deals with simple automorphic loops.

Problem 4.2. *Does there exist a nonassociative finite simple automorphic loop?*

In the commutative case, it has been shown that no finite simple nonassociative commutative automorphic loop exists [11]. The proof used connections between commutative automorphic loops and both Bruck loops and Lie algebras.

In general, using a computational approach, we know there is no nonassociative simple automorphic loop of order less than 2500 [16]. We can, however, reduce this problem to a purely group theoretical question.

From the work of Niemenmma and Kepka, [23], we can state explicitly when a group G is the multiplication group of a loop Q.

Theorem 4.3. *([23]). A group G is the multiplication group of a loop* if and only if *there is a subgroup H and transversals R, L of H such that*

(i) $G = \langle R, L \rangle$,
(ii) H is corefree, that is, $\bigcap_{g \in G} H^g = 1$,
(iii) $[R^{-1}, L^{-1}] \leq H$.

We can specialize this to automorphic loops as follows:

Theorem 4.4. *([16]). In the setting of Theorem 4.3, the corresponding loop will be automorphic* if and only if $R^h = R$ *and* $L^h = L$ *for all* $h \in H$.

Finally, Albert show that if a loop Q is simple *if and only if* its multiplication group $\mathrm{Mlt}(Q)$ is primitive on Q [1]. Hence, we can restate Problem 4.2 as follows:

Problem 4.5. *Is there a permutation group G and two subsets $R, L \subseteq G$ containing 1_G such that:*

(a) G is primitive,
(b) R and L are right and left transversals to $H = G_1$ in G,
(c) $G = \langle R, L\rangle$,
(d) $[R^{-1}, L^{-1}] \leq H$,
(e) $R^h = R$ and $L^h = L$ for every $h \in H$?

Since our question depends on primitive permutation groups, can make use of the O'Nan-Scott Theorem on classifying primitive permutation groups. Recalling the *socle* $\mathrm{Soc}(G)$ of a group G, we can divide our study of finite primitive groups into two cases, depending on whether the $\mathrm{Soc}(G)$ is regular or not.

Theorem 4.6. *([19]). Let Q be a finite simple nonassocitive automorphic loop. Then the* $\mathrm{Soc}(\mathrm{Mlt}(Q))$ *is not regular.*

By the O'Nan-Scott Theorem, it follows that $\mathrm{Mlt}(Q)$ is of product type, of diagonal type or of almost simple type.

4.2. **Metabelian groups, commutative automorphic loops and Γ-loops.** Returning to $(G, \circ)$ in §3.1, if G is nilpotent of class at most 2, then $(G, \circ)$ is an abelian group. In this case, the passage from G to $(G, \circ)$ is called the "Baer trick" [12]. Since commutative automorphic loops can be seen as one generalization of abelian groups, a natural question to ask is when $(G, \circ)$ is a commutative automorphic loop? If G is 2-Engel, then $(G, \circ)$ is Moufang (and in fact coincides with $(G, \oplus)$), and if G is nilpotent of class at most 3, then $(G, \circ)$ is a Γ-loop of nilpotent class 2. Summarizing:

Group Property	**Γ-Loop Description**	**Proved**
nilpotent class 2	abelian group	✓
2−Engel	commutative Moufang loop	✓
nilpotent class 3	Γ-loop of nilpotent class 2	✓
metabelian	commutative automorphic loop	

When G is metabelian, $(G, \circ)$ is conjectured to be a commutative automorphic loop. Our only examples of a group G with $(G, \circ)$ not a commutative automorphic loop occur when G is nonmetabelian.

Problem 4.7. *Let G be a uniquely 2-divisible metabelian group. Is $(G, \circ)$ a commutative automorphic loop?*

Moreover, using the above remarks, we can ask:

Problem 4.8. *Let $(Q, \circ)$ be a commutative automorphic loop and let $(Q, \oplus)$ be the corresponding Bruck loop. Is the left multiplication group of $(Q, \oplus)$ metabelian?*

We have two main approaches to this problem. Since our early results are purely based in commutator calculus, we wish to study nilpotent and metabelian groups in depth. Since minimal properties of 2-Engel groups show $(G, \circ)$ is Moufang, it is natural to ask what metabelian properties prove that $(G, \circ)$ is a commutative automorphic loop? Secondly, in the 2-Engel case, $(G, \circ) = (G, \oplus)$, and $(G, \oplus)$ has a geometrical interpretation. Perhaps our understanding of $(G, \circ)$ would be deeper if we can find a reasonable geometric interpretation of it.

4.3. **Simple Semiautomorphic IP loops.** Paige gave a construction for simple Moufang loops using a field $\mathbb{F}$, R, the set of matrices $\left(\begin{smallmatrix} a & \alpha \\ \beta & b \end{smallmatrix}\right)$ where α, β are 3-dimensional coordinate vectors over $\mathbb{F}$, and Zorn's multiplication

$$\left(\begin{smallmatrix} a & \alpha \\ \beta & b \end{smallmatrix}\right) \star \left(\begin{smallmatrix} c & \gamma \\ \delta & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} ac+\alpha\circ\delta & a\gamma+\alpha d-\beta\times\delta \\ \beta c+b\delta+\alpha\times\gamma & \beta\circ\gamma+bd \end{smallmatrix}\right)$$

with $\alpha \circ \beta, \alpha \times \beta$ are scalar and vector products. Then

Theorem 4.9. *([24]). $(R, \star)$ is a Moufang loop. Moreover, if R is restricted to matrices where $ab - \alpha \circ \beta = 1$, then $(R, \star)/Z(R, \star)$ is a simple Moufang loop.*

Liebeck was able to show that all finite simple Moufang loops have the form above [21]. From §3.2, starting with a nonabelian group, we can construct a Moufang loop and a semiautomorphic IP loop. Moreover, much of the structure in Moufang loops continues in semiautomorphic IP loops. Hence, it is natural to ask if we can construct simple semiautomorphic IP loops in a manner similar as Paige did. That is,

Problem 4.10. *What alterations to $(R, \star)$ can be made to have $(R, \star)/Z(R, \star)$ be a simple, non Moufang, semiautomorphic IP loop?*

References

1. A.A. Albert, *Quasigroups, I*, Trans. Amer. Math. Soc. **54** (1943), 507–519.
2. A.S. Basarab, *Klass LK-lup*, Matematicheskie issledovanija **120** (1991), 3–7.
3. R.H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, Berlin, 1971.
4. R.H. Bruck and L.J. Paige, *Loops whose inner mappings are automorphisms*, Ann. of Math. (2) **2** (1956), 308–323.
5. D.A.S. de Barros, A. Grishkov, and P. Vojtěchovský, *Commutative automorphic loops of order p^3*, LMS Journal of Computation and Mathematics (to appear).
6. A. Drapal, *On multiplication groups of left conjugacy closed loops*, Comment. Math. Univ. Carolinae **45** (2004), 223–236.
7. G. Glauberman, *On loops of Odd Order I*, Journal of Algebra **1** (1964), 374–396.
8. ______, *On loops of Odd Order II*, Journal of Algebra **8** (1968), 393–414.
9. M. Greer, *A class of loops categorically isomorphic to uniquely* 2*-divisible bruck loops*, (submitted).
10. ______, *Semiautomorphic inverse property loops*, (submitted).
11. A. Grishkov, M. Kinyon, and G. Nagy, *Solvability of commutative automorphic loops*, Proc. Amer. Math. Soc. (to appear).
12. I.M. Isaacs, *Finite group theory*, vol. 92, Graduate Studies in Mathematics, AMS, 2008.
13. P. Jedlička, M.K. Kinyon, and P. Vojtěchovský, *Constructions of commutative automorphic loops*, Comm. Algebra **38** (2010), 3243–3267.
14. ______, *The structure of commutative automorphic loops*, Trans. Amer. Math. Soc. **363** (2011), 365–384.
15. ______, *Nilpotency in automorphic loops of prime power order*, Journal of Algebra **350** (2012), 64–76.

16. K.W. Johnson, M.K. Kinyon, G.P. Nagy, and P. Vojtěchovský, *Searching for small simple automorphic loops*, LMS Journal of Computation and Mathematics **14** (2011), 200–213.
17. H. Kiechle, *Theory of K-loops*, vol. 1778, Lecture Notes in Mathematics, Springer, 2008.
18. M.K. Kinyon, K. Kunen, and J.D. Phillips, *A generalization of Moufang and Steiner loops*, Algebra Universalis **48** (2002), 81–101.
19. M.K. Kinyon, K. Kunen, J.D. Phillips, and P. Vojtěchovský, *The structure of automorphic loops*, available at arXiv.1210.1642.
20. M.K. Kinyon and P. Vojtěchovský, *Primary decompositions in varieties of commutative diassociative loops*, Communications in Algebra **37** (2009), 1428–1444.
21. M. Liebeck, *The classification of finite simple Moufang loops*, Math. Proc. Cambridge Phil, Soc. **102** (1997), 33–47.
22. P. Nagy and K. Strambach, *Loops as invariant sections in groups, and their geometry*, Canad. J. Math. **46** (1994), 1027–1056.
23. M. Niemenmaa and T. Kepka, *On multiplication groups of loops*, Journal of Algebra **135** (1990), 112–122.
24. L.J. Paige, *A class of simple Moufang loops*, Proc. Amer. Math. Soc. **7** (1956), 471–482.
25. H.O. Pflugfelder, *Quasigroups and Loops: Introduction*, Sigma Series in Pure Math, Heldermann Verlag, Berlin, 1990.
26. A.A. Ungar, *The relativistic noncommutative nonassociative group of velocities and the Thomas rotation*, Resultate Math **16** (1989), 168–179.